$$\Leftrightarrow 4n\left(\sum a\right)^2 \ge (12n+1)\sum ab - \sum a^2 \Rightarrow (4n+1)\left(\sum a^2 - \sum ab\right) \ge 0$$

that is true.

Nicuşor Zlota

W11. (Solution by the proposer.) First, we observe that there are numbers satisfying the equality stated, for instance, $1, 2, 4$, and 5. To prove the statement we argue by contradiction. Suppose that there exists four positive integers x, y, z, t with $x > y > z > t$ such that $x^2 + xz - z^2 = y^2 + yt - t^2$ and $p = xy + zt$ is prime. Substituting $x = \dfrac{p - zt}{y}$ in the expression $x^2 + xz - z^2 = y^2 + yt - t^2$, we get

$$\left(\frac{p - zt}{y}\right)^2 + \left(\frac{p - zt}{y}\right) z - z^2 = y^2 + yt - t^2$$

and reordering terms, yields

$$p(p - 2zt + yz) = (y^2 + z^2)(y^2 + yt - t^2)$$

Since p is prime, then p divides $y^2 + z^2$ or divides $y^2 + yt - t^2$.

- If $p \mid y^2 + z^2$, then $0 < y^2 + z^2 < 2xy < 2(xy + zt) = 2p$ and this implies $y^2 + z^2 = p = xy + zt$ from which follows $y \mid z(z - t)$. Since $xy + zt$ is prime, then $\gcd(y, z) = 1$, and therefore, $p \mid (z - t)$ which is impossible because $0 < z - t < z < y$.

- If $p \mid y^2 + yt - t^2$, then $0 < y^2 + yt - t^2 < 2(xy + zt) = 2p$ and this implies $y^2 + yt - t^2 = p$. That is, $xy + zt = y^2 + yt - t^2 = x^2 - xz - z^2$ from which follows $x \mid z(z + t)$ and $y \mid t(z + t)$. As $\gcd(xy, zt) = 1$, then $x \mid (z + t)$ and $y \mid (z + t)$. Since $0 < z + t < 2x$ and $0 < z + t < 2y$, then $z + t = x$ and $z + t = y$ which is impossible.

The preceding contradictions let us to conclude that $xy + zt$ is composite.

W12. (Solution by the proposer.) The limit equals $4e^{\gamma}$ where γ denotes the Euler-Mascheroni constant. A calculation shows that

$$O_n = \gamma_{2n} - \frac{1}{2}\gamma_n + \ln 2 + \ln \sqrt{n},$$

where $\gamma_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln n$.
Let $y_n = \frac{2O_n}{n}$, $n \geq 1$. We have

$$x_n = \frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n = \frac{1}{n}e^{\ln(1+x_n)} = \frac{1}{n}e^{n\left(\frac{\ln(1+y_n)-y_n}{y_n^2}\cdot y_n^2+y_n\right)} =$$

$$= \frac{1}{n}e^{\frac{\ln(1+y_n)-y_n}{y_n^2}\cdot ny_n^2+ny_n} = \frac{1}{n}e^{\frac{\ln(1+y_n)-y_n}{y_n^2}\cdot ny_n^2} \cdot e^{ny_n}$$

We have $\lim\limits_{n\to\infty} ny_n^2 = 0$ and, since $\lim\limits_{n\to\infty} y_n = 0$, we get that $\lim\limits_{n\to\infty} \frac{\ln(1+y_n)-y_n}{y_n^2} = -\frac{1}{2}$. On the other hand,

$$ny_n = 2O_n = 2\gamma_{2n} - \gamma_n + \ln 4 + \ln n$$

and it follows that

$$x_n = e^{\frac{\ln(1+y_n)-y_n}{y_n^2}\cdot ny_n^2} \cdot e^{2\gamma_{2n}-\gamma_n+\ln 4}$$

which implies $\lim\limits_{n\to\infty} x_n = 4e^{\gamma}$.

Second solution. Let $h_n := \sum\limits_{k=1}^{n} \frac{1}{k}$ and $\alpha_n := \frac{2O_n}{n}$. Since

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n = \lim_{n\to\infty} \frac{1}{n}(1+\alpha_n)^n = \lim_{n\to\infty} e^{n\ln(1+\alpha_n)-\ln n}$$

we will find

$$\lim_{n\to\infty} (n\ln(1+\alpha_n) - \ln n).$$

Noting that $O_n = h_{2n} - \frac{1}{2}h_n$ and $\ln n + \gamma < h_n < \ln(n+1) + \gamma$, where $\gamma \approx 0.5772...$ is Euler's constant we obtain

$$2(\ln(2n) + \gamma) - (\ln(n+1) + \gamma) < 2O_n < 2(\ln(2n+1) + \gamma) - (\ln n + \gamma) \iff$$

$$\iff \ln\frac{4n^2}{n+1} + \gamma < 2O_n < \ln\frac{(2n+1)^2}{n} + \gamma \iff$$

$$\iff \frac{1}{n}\left(\ln\frac{4n^2}{n+1} + \gamma\right) < \alpha_n < \frac{1}{n}\left(\ln\frac{(2n+1)^2}{n} + \gamma\right) \qquad (1)$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\left(\ln\frac{4n^2}{n+1}+\gamma\right)=\lim_{n\to\infty}\frac{1}{n}\left(\ln\frac{(2n+1)^2}{n}+\gamma\right)=0$$

then by Squeeze Principle $\lim\limits_{n\to\infty}\alpha_n=0$.

Furthermore, $\lim\limits_{n\to\infty}n\alpha_n^2=0$ as well. Indeed, since

$$\frac{1}{n}\left(\gamma+\ln\frac{4n^2}{n+1}\right)^2<n\alpha_n^2<\frac{1}{n}\left(\gamma+\ln\frac{(2n+1)^2}{n}\right)^2$$

and

$$\lim_{n\to\infty}\frac{\gamma+\ln\dfrac{4n^2}{n+1}}{\ln 4n}=\lim_{n\to\infty}\frac{\gamma+\ln\dfrac{(2n+1)^2}{n}}{\ln 4n}=1$$

then

$$\lim_{n\to\infty}\frac{1}{n}\left(\gamma+\ln\frac{4n^2}{n+1}\right)^2=\lim_{n\to\infty}\frac{1}{n}\left(\gamma+\ln\frac{(2n+1)^2}{n}\right)^2=$$

$$=\lim_{n\to\infty}\frac{\ln^2 4n}{n}=\lim_{n\to\infty}\left(\frac{\ln 4n}{\sqrt{n}}\right)^2=\left(\lim_{n\to\infty}\frac{\ln n+\ln 4}{\sqrt{n}}\right)^2=\left(\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}\right)^2=0.$$

Using inequalities $x-\dfrac{x^2}{2}<\ln(1+x)<x, x>0$ and **(1)** we obtain

$$\ln\frac{4n^2}{n(n+1)}+\gamma-\frac{n\alpha_n^2}{2}<n\alpha_n-\frac{n\alpha_n^2}{2}-\ln n<\underline{n\ln(1+\alpha_n)-\ln n}<$$

$$<n\alpha_n-\ln n<\ln\frac{(2n+1)^2}{n^2}+\gamma\implies\ln\frac{4n^2}{n(n+1)}+\gamma-\frac{n\alpha_n^2}{2}<$$

$$<n\ln(1+\alpha_n)-\ln n<\ln\frac{(2n+1)^2}{n^2}+\gamma.$$

Since

$$\lim_{n\to\infty}\ln\frac{4n^2}{n(n+1)}=\lim_{n\to\infty}\ln\frac{(2n+1)^2}{n^2}=\ln 4$$

and

$$\lim_{n\to\infty} n\alpha_n^2 = 0$$

then by Squeeze Principle we obtain

$$\lim_{n\to\infty} (n \ln(1+\alpha_n) - \ln n) = \ln 4 + \gamma$$

and, therefore,

$$\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{2O_n}{n}\right)^n = \lim_{n\to\infty} e^{\ln 4+\gamma} = 4e^{\gamma}.$$

Arkady Alt

Third solution. Let $h_n = \sum_{k=1}^{n} \frac{1}{k}$. Since $O_n = h_{2n} - \frac{1}{2}h_n$ and
$\ln n + \gamma < h_n < \ln(n+1) + \gamma$, where
$\gamma \approx 0.5772...$ is Euler's constant 6 then

$$\ln(2n) + \gamma - \frac{1}{2}(\ln(n+1)+\gamma) < O_n < \ln(2n+1) + \gamma - \frac{1}{2}(\ln n + \gamma) \iff$$

$$\iff \ln\left(\frac{2n}{\sqrt{n+1}}\right) + \frac{\gamma}{2} < O_n < \ln\left(\frac{2n+1}{\sqrt{n}}\right) + \frac{\gamma}{2} \iff$$

$$\iff \ln\left(\frac{2n}{2\sqrt{n}\sqrt{n+1}}\right) < \alpha_n < \ln\left(\frac{2n+1}{2n}\right),$$

where

$$\alpha_n := O_n - \ln\left(2\sqrt{n}\right) - \frac{\gamma}{2} = O_n - \frac{1}{2}\ln n - \frac{\gamma}{2} - \ln 2.$$

Hence $O_n = \frac{1}{2}\ln n + \frac{\gamma}{2} + \ln 2 + \alpha_n, n \in \mathbb{N}$ and by Squeeze Principle
$\lim_{n\to\infty} \alpha_n = \ln 1 = 0.$
Note that

$$\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{2O_n}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{2O_n}{n}\right)-\ln n}.$$

Since $\ln(1+x) = x - \frac{x^2}{2} + o\left(x^2\right), x > 0$ then

$$n \ln\left(1+\frac{2O_n}{n}\right) - \ln n = n\left(\frac{2O_n}{n} + \frac{4O_n^2}{2n^2} + o\left(\frac{O_n^2}{n^2}\right)\right) =$$

$$= 2O_n + \frac{2O_n^2}{n} + o\left(\frac{O_n^2}{n}\right) - \ln n = \ln n + 2\ln 2 + \gamma + 2\alpha_n + \frac{2O_n^2}{n} + o\left(\frac{O_n^2}{n}\right) - \ln n =$$

$$= 2\ln 2 + \gamma + 2\alpha_n + \frac{2O_n^2}{n} + o\left(\frac{O_n^2}{n}\right)$$

and, thertefore

$$\lim_{n\to\infty}\left(n\ln\left(1+\frac{2O_n}{n}\right) - \ln n\right) = 2\ln 2 + \gamma + \lim_{n\to\infty}\left(2\alpha_n + \frac{2O_n^2}{n} + o\left(\frac{O_n^2}{n}\right)\right) =$$

$$= 2\ln 2 + \gamma + \lim_{n\to\infty}\left(\frac{2O_n^2}{n} + o\left(\frac{O_n^2}{n}\right)\right) = 2\ln 2 + \gamma$$

because

$$\lim_{n\to\infty}\frac{O_n^2}{n} = \lim_{n\to\infty}\frac{\ln^2 n}{n} = 0 \implies \lim_{n\to\infty} o\left(\frac{O_n^2}{n}\right) = 0.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n}\left(1+\frac{2O_n}{n}\right)^n = \lim_{n\to\infty} e^{2\ln 2+\gamma} = 4e^{\gamma}.$$

Arkady Alt

W13. (Solution by the proposer.) The limit equals

$$\begin{cases} e^l, \ p = 1 \\ e^{l+2}, \ p = 2 \\ \infty, \ p \geq 3 \end{cases}$$

We have that

$$x_n = \sum_{k=1}^{n} \ln\left(a_n + \frac{1}{\sqrt[p]{kn}}\right) = \sum_{k=1}^{n} \ln\left(1 + \left(a_n - 1 + \frac{1}{\sqrt[p]{kn}}\right)\right) =$$

$$= \sum_{k=1}^{n} \frac{\ln\left(1+\left(a_n - 1 + \frac{1}{\sqrt[p]{kn}}\right)\right)}{a_n - 1 + \frac{1}{\sqrt[p]{kn}}}\left(a_n - 1 + \frac{1}{\sqrt[p]{kn}}\right)$$